

# The non-local repercussions of partial jamming in glassy and granular flows

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This Letter establishes a link between the non-local behaviour of glassy materials and the presence of transient clusters of jammed particles within the flow. These clusters are first evidenced in simulated dense granular flows subjected to plane shear, and are found to originate from a mechanism of multiple orthogonal shear banding. A continuum non-local model, similar in form to the non-local Kinetic-Elasto-Plastic model, is then directly derived by simply considering the spatial redistribution of vorticity induced by these clusters. The non-locality length scale is thus found to be directly related to the cluster size. The finding of this purely kinematic origin indicates that non-local behaviour should be expected in all glassy materials, regardless of their local constitutive law, as long as they partially jam during flow.

Glassy materials such as foams, emulsions, and granular matter are composed of many particles, be they bubbles, droplets or grains, that interact with their neighbours upon deformation. These interactions underpin rich flow behaviours that are pivotal to a number of application in engineering, geophysics and biophysics.

Different local constitutive laws were established to predict the flow of glassy materials, including Herschel-Bulkley model for foams and emulsions [1] and viscoplastic model for granular materials [2]. However, strong deviations from these laws were evidenced in flow geometries involving the proximity of walls [3, 4] and/or stress gradients [5]. These deviations were attributed to non-local behaviours, and a range of non-local models were introduced to capture them [3, 6–9]. Amongst them, a non-local kinetic elasto plastic model (KEP) [7] was shown to successfully predict the flow properties of both emulsions [6] and granular materials [10] in many geometries. Intriguingly, these materials satisfy two different local constitutive laws. This suggests that non-local behaviours may be independent from local behaviours. It also implies that there should exist a common mechanism governing non-locality for both emulsions and granular materials, and possibly for other similar materials.

In this Letter we establish the link between non-local behaviours and the presence of transient clusters of jammed particles within the flow. Clusters are first identified and characterised in dense granular flows. A non-local model similar to the KEP model is then derived based on a purely kinematic argument considering the way clusters spatially redistribute the local shear rate.

*Internal length scale in dense granular flows*—Beyond their difference in formulation, all existing non-local models involve a typical length scale  $\xi$  representing the spatial extent of non-locality. For granular materials, this length was found to scale with the inertial number  $I$  as:

$$\xi \propto \frac{d}{I^\alpha}, \text{ with } I = \dot{\gamma} d \sqrt{\frac{\rho}{P}}; \quad (1)$$

where  $d$  and  $\rho$  are the diameter and the density of the grains,  $P$  and  $\dot{\gamma}$  are the normal stress and the shear strain rate the material is subjected to. The value of the exponent was found to be  $\alpha = 0.5$  for frictional granular flows [4, 9, 10] and  $\alpha = 0.25$  for frictionless grains mimicking droplets in emulsions [7, 9, 11]. However, the microscopic origin of this length scale remains conjectural.

In order to identify what property of the microstructure could govern the non-local length scale, we performed discrete element simulations of dense granular flows in which the motion of each grain, both translation and rotation, is integrated over small time steps using a second-order predictor corrector scheme, as in [4]. A system of 10 000 grains in a 2D periodic domain is subjected to shear, prescribing both the shear rate  $\dot{\gamma}$  and the normal stress  $P$  (see Fig. 1(a)). Grains have a polydispersity

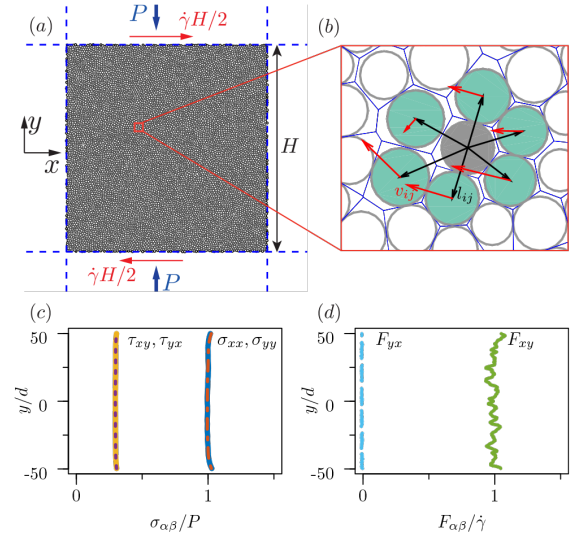


FIG. 1. Homogeneous shear of granular materials. (a) 100×100 grains sheared with bi-periodic boundary condition; (b) Close up of a grain (grey) and its Voronoi neighbours (green); (c-d) Time average profiles of stresses and velocity gradients for a system with  $I = 0.01$ .

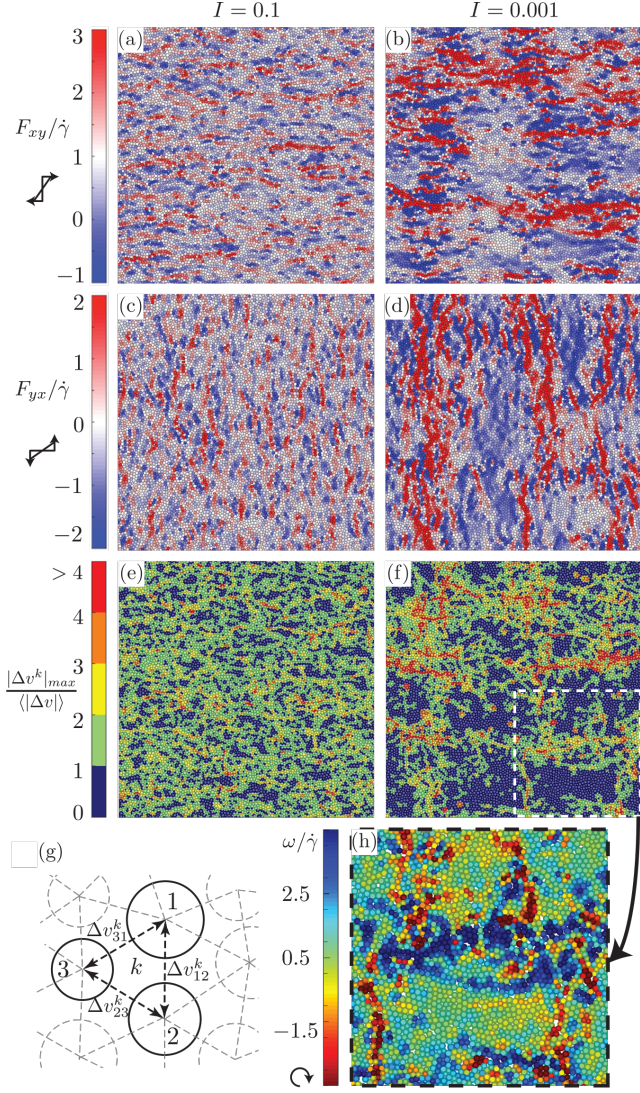


FIG. 2. Multiple shear bands and clusters of jammed grains for two inertial numbers. (a-d) First two rows are snapshots of spatial distribution of the velocity gradients along  $x$  direction and  $y$  direction respectively. (e-f) Snapshots of maximum relative velocities for each triplet of grains (see text); Jammed triplets are shown in dark blue. (g) Schematic of Delaunay triangulation and computation of relative velocities for triplet of grains. (h) A closeup of (f) showing the local vorticity  $\omega = \frac{1}{2}(F_{xy} - F_{yx})$ .

of  $\pm 20\%$  on their diameter  $d$  in order to avoid crystallisation. They interact with their neighbours *via* inelastic and frictional contacts, characterised by a Young's modulus of  $E = 10^3 P$ , a coefficient of restitution  $e = 0.5$  and a coefficient of friction  $\mu_g = 0.5$ , or  $\mu_g = 0$  corresponding to frictionless grains.

The advantage of the plane shear geometry is to produce homogeneous steady flows in which stresses and shear rate are constant throughout the shear layer (see Fig. 1(c-d)). The use of Lees-Ewards periodic boundary conditions [12] prevents the introduction of walls

and avoids the shear heterogeneity they would induce [4]. Such homogeneous shear flows are necessary for finding the microstructural origin of  $\xi$ , as this length is expected to vary with both the shear rate and the normal stress according to (1). Several steady flows of differing shear rate  $\dot{\gamma}$  were performed within the dense regime [2],  $5.10^{-4} \leq I \leq 10^{-1}$ .

*Mechanism of cluster formation*—Figure 2(a-d) show a snapshot of the local velocity gradients within two flows with different inertial numbers. Local velocity gradients are quantified by the tensor  $\mathbf{F}^i$ , which is defined for each grain  $i$  by considering its relative velocity  $\mathbf{v}^{i,j}$  and distance  $\mathbf{l}^{i,j}$  to its Voronoi neighbours  $j$  (as shown in figure 1c) by employing the formula [4]:  $\mathbf{F}_{\alpha\beta}^i \equiv \left\langle \frac{\partial v_\alpha}{\partial x_\beta} \right\rangle_i = \langle \mathbf{l}^{i,j} \otimes \mathbf{l}^{i,j} \rangle^{-1} \bullet \langle \mathbf{l}^{i,j} \otimes \mathbf{v}^{i,j} \rangle$ , where  $\bullet$  and  $\otimes$  represent the tensor product and outer product, and  $\langle \cdot \rangle$  represents the average over all  $j$  neighbours of grain  $i$ . The components  $F_{yx}$  and  $F_{xy}$  shown on figure 2(a-d) denote a rate of shear deformation parallel and orthogonal to the flow direction  $x$ , respectively. It appears that the shear is localised on multiple shear bands both in the flow direction and in the transverse direction. These two directions correspond to the direction of maximum shear stress in the flow, given that the two normal stresses are equal (see Fig. 1(c)). This mechanism of multiple and orthogonal shear banding creates a lattice of highly sheared zones delimiting the boundaries of cluster of grains subjected to little if any shear deformation. Figure 2(h) shows that the local vorticity is nearly constant within clusters with a value close to  $\dot{\gamma}/2$ , further evidencing that clusters rotate like rigid bodies.

*Measuring the size of clusters*—As a way to identify individual clusters and measure their size, triplets of neighbouring grains are identified using a Delaunay triangulation and a criteria is employed to determine which triplets are jammed. Amongst several possibilities, we used a criteria based on the maximum relative velocity between pair of grains in the triplet  $k$ :  $|\Delta v^k|_{max} = \max(|\Delta v_{12}^k|; |\Delta v_{13}^k|; |\Delta v_{23}^k|)$  (see Fig. 2 (g)). The triplet is considered jammed if all of the three relative velocities are smaller than the average relative velocity  $\langle |\Delta v| \rangle$  between all pair of neighbouring grains in the full system at a given snapshot.

Such jammed triplets are shown in dark blue in figures 2 (e-f). Finally, an aggregating algorithm is used to connect adjacent jammed triplets into clusters. The measured size  $\ell_m$  of a cluster of  $n$  grains is then defined as  $\ell = d\sqrt{n}$ .

Figure 3 shows the cluster size distribution obtained by using the above criteria on 1000 snapshots evenly distributed over a total shear deformation of 50. The PDF follows a power law with an exponential cutoff. The value of this cutoff increases for decreasing inertial number, showing that larger clusters are more likely to appear in systems with smaller inertial numbers.

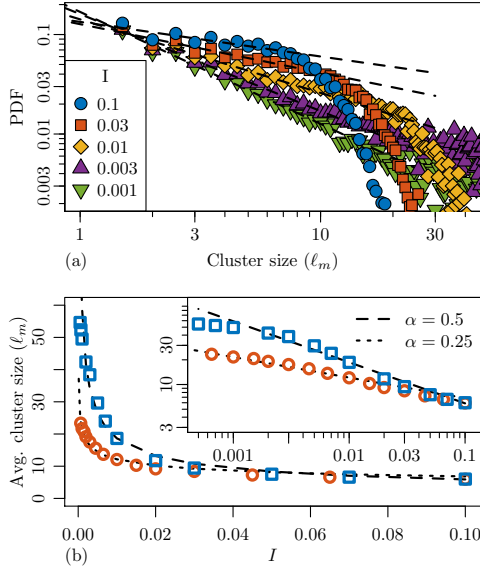


FIG. 3. Measured cluster size. (a) Probability distribution of the cluster size for frictional granular flows. (b) The scaling of average cluster size for frictional ( $\square$ ) and frictionless ( $\circ$ ) granular flows. Inset: Same data in log-log scale. The curves represent power law fit,  $\ell \propto d/I^\alpha$ , which is similar to (1).

Strikingly, the average cluster size  $\ell_m$  scales with the inertial number with a power law which is similar to the scaling of the non-local length  $\xi$  in (1). This suggests that jammed clusters could be a microstructural origin of non-locality.

*Link between clusters and non-locality*—We now seek to establish how the existence of jammed clusters can give rise to non-local behaviours at the continuum scale. The key consideration is that clusters move as rigid bodies, and therefore redistribute their vorticity over their size (see Fig. 2 (h)). The local value of the vorticity in a flow should therefore be affected by the value of the vorticity in the surrounding. In the following a non-local continuum model is derived based on this mechanism of vorticity redistribution.

Within a homogeneously sheared layer subjected to a bulk shear rate  $F_{xy} = \dot{\gamma}_b$  and  $F_{yx} = 0$  (see Fig. 1 (d)), the bulk shear rate  $\dot{\gamma}_b$  will depend on the state of stress and it can be back calculated using the local constitutive law of the material, for instance the  $\tau/P = \mu(I)$  law for granular materials [2]. This deformation can be decomposed into a pure shear deformation,  $\dot{\epsilon}_b = \dot{\gamma}_b/2$ , and pure rotation represented by the vorticity,  $\omega_b = \dot{\gamma}_b/2$ :  $\dot{\gamma}_b = \dot{\epsilon}_b + \omega_b$ . The shear deformation is responsible for local mechanical dissipation, and the vorticity governs the cluster rotation rate which gets redistributed.

This decomposition also applies locally in a non-homogeneously sheared layer in which the shear rate depends on  $y$ :  $\dot{\gamma}(y) = \dot{\epsilon}(y) + \omega(y)$ . The velocity gradient

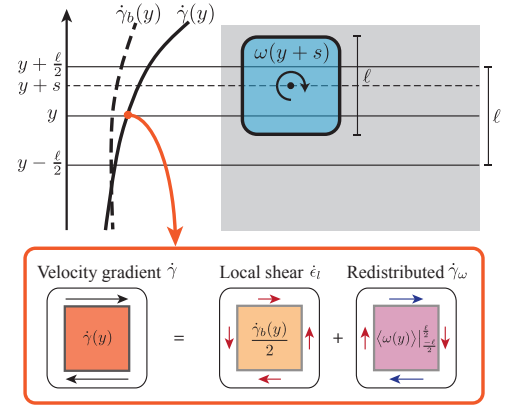


FIG. 4. Illustration of the proposed mechanism at the origin of non-locality. (Top) Actual shear rate profile  $\dot{\gamma}(y)$  in a heterogeneous sheared layer compared to the shear rate profile  $\dot{\gamma}_b$  predicted by a local constitutive law with no account for non-local effects; A clusters of jammed particles located at a position  $y + s$  is represented, distributing its vorticity over its size  $\ell$ . (bottom) The local velocity gradient is comprised of i) a local pure shear strain rate  $\dot{\epsilon} = \dot{\gamma}_b/2$  governed by the local stresses via a local constitutive law, and a contribution  $\dot{\gamma}_\omega = \langle \omega(y) \rangle|_{-\ell/2}^{\ell/2}$  coming from the vorticities of nearby clusters.

can also be decomposed into a local shear strain deformation that is not redistributed,  $\dot{\epsilon}_l$ , and a shear rate due to redistribution of vorticity,  $\dot{\gamma}_\omega$ :

$$\dot{\gamma}(y) = \dot{\epsilon}_l(y) + \dot{\gamma}_\omega(y) \quad (2)$$

In the homogeneous case, the vorticity is constant throughout all layers and the shear rate due to redistribution of vorticity  $\dot{\gamma}_\omega$  is therefore equal to the vorticity  $\omega$  and will contain only the rotational component. By contrast, in the non-homogeneous case, this may not be the case and  $\dot{\gamma}_\omega$  may contain some pure shear component along with rotation depending on the geometry, as illustrated in Fig 4.

When a cluster of size  $\ell$  develops at some position  $y + s$ , it redistributes the vorticity  $\omega(y + s) = \dot{\gamma}(y + s)/2$  over the zone comprised over its size  $\ell$ . Given that these clusters are transient and develop at different locations, the net shear rate at a point  $y$  due to redistribution of vorticity is given by the average of the vorticities of points between  $y \pm \ell/2$ , i.e.,  $\dot{\gamma}_\omega(y) = \langle \omega(y) \rangle|_{-\ell/2}^{\ell/2} = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} \dot{\gamma}(y + s)/2 ds$ . By contrast, jammed clusters do not redistribute local pure shear, which is entirely govern by the local stresses in the layer and is therefore the same as in the case of homogeneous shear flow, i.e.,  $\dot{\epsilon}_l(y) = \dot{\epsilon}_b(y) = \dot{\gamma}_b(y)/2$ . Substituting these two results in (2) leads to:

$$2\dot{\gamma}(y) = \dot{\gamma}_b(y) + \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} \dot{\gamma}(y + s) ds \quad (3)$$

By using a Taylor expansion of  $\dot{\gamma}(y + s)$  with respect to  $s$  about  $s = 0$ , the integral in (3) becomes:



$$\begin{aligned} \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} \left( \dot{\gamma}(y) + \frac{\partial \dot{\gamma}(y)}{\partial y} s + \frac{1}{2} \frac{\partial^2 \dot{\gamma}(y)}{\partial y^2} s^2 + \mathcal{O}(s^3) \right) ds \\ = \dot{\gamma}(y) + \frac{\ell^2}{24} \frac{\partial^2 \dot{\gamma}(y)}{\partial y^2} + \mathcal{O}(\ell^4) \end{aligned}$$

Introducing the second order approximation of this expression into (3) leads to a non-local equation governing the shear rate  $\dot{\gamma}(y)$ :

$$\dot{\gamma}(y) - \dot{\gamma}_b(y) = \frac{\ell^2}{24} \frac{\partial^2 \dot{\gamma}}{\partial y^2}. \quad (4)$$

This expression is similar to the non-local KEP model which is usually written in terms of *fluidity*  $f$  [7]:  $f(y) - f_b(y) = \xi^2 \nabla^2 f$ , with  $\xi$  the so called *cooperativity* length. Two different definitions were used for the fluidity,  $f = \dot{\gamma}/\tau$  for emulsions [6] and  $f = \dot{\gamma}/\mu$  for granular materials [10]. In both cases, however, non-local effects arise when flows are near the jamming transition, in a flow regime where the shear stress  $\tau$  for emulsions and friction coefficient  $\mu$  for granular materials are nearly constant. The formulation in terms of fluidity then reduces to (4) and the cooperativity length is directly given by the average cluster size:

$$\xi = \ell / \sqrt{24} \quad (5)$$

As a way to assess the validity of the non-local model (4), let us now consider a series of granular flows in a Poiseuille geometry (see Fig. 5(a)). This geometry involves some stress gradient in the  $y$  direction and the presence of fixed walls, which both induce non-local effects [4, 10]. Figure 5(c) shows the measured size of clusters  $\ell_m(y)$ , which appears to be smaller near the fixed walls. Figures 5(b,d) show that the model (4) is able to reproduce all the measured velocity and shear rate profiles by using the cluster size  $\ell(y) = A\ell_m(y)$ , where  $A = 0.39$  is a fitting constant.

**Conclusions**—The key result of this Letter is that a non-local continuum model (4) can be derived from a purely kinematic argument, considering the existence of jammed clusters in the flow and the way they spatially redistribute the vorticity. The cooperativity length scale  $\xi$ , the only parameter of this model, is further found to be related to the average cluster size according to (5), which itself depends on the inertial number  $I$  with a power law (Fig.3).

Convective motion of particles[13], and avalanche-like rearrangements have been observed in various glassy materials like sand [14], granular materials [15], foams [16], suspensions [6, 17], colloids[18], Lennard-Jones glass [19]. Our results imply that non-local behaviours should be expected in all such glassy materials so long as jammed clusters develop within the flow. Subsequently, it indicates that non-local behaviours originate from and are governed by a common mechanism that is independent

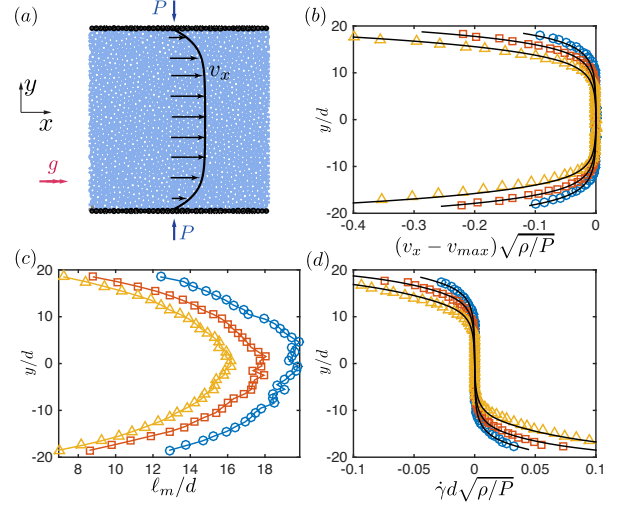


FIG. 5. Non-locality and clusters in a heterogeneously sheared granular flow. (a) Poiseuille flow geometry where walls grains (black) are stationary and flowing grains (blue) are subjected to a constant force of  $mg$  along the  $x$  direction. (b) Velocity profile, (c) Measured cluster size, and (d) shear-rate profile for systems subjected to  $mg/(d^2 P) = 0.016$  ( $\circ$ ),  $0.018$  ( $\square$ ), and  $0.022$  ( $\triangle$ ). Solid lines in (b) and (c) are solution of equation (4) using  $\ell(y) = A\ell_m(y)$ , where  $A$  is a constant.

from the specific nature of the material local constitutive law. This suggests that non-locality should be a universal feature of glassy materials and could therefore be described by a unique model such as (4) taking into account the particular cluster size that develops in specific materials.

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